Tikrit university
College of Engineering
Mechanical Engineering Department

Lectures on Engineering Analysis

Chapter 4
Fourier Series

Assistant Prof. Dr. Eng. Ibrahim Thamer Nazzal

Assistant Prof. Dr. Ibrahim Thamer Nazzal

Fourier Series

When the French mathematician Joseph Fourier (1768–1830) was trying to solve a problem in heat conduction, he needed to express a function as an infinite series of sine and cosine functions which named Fourier series later:

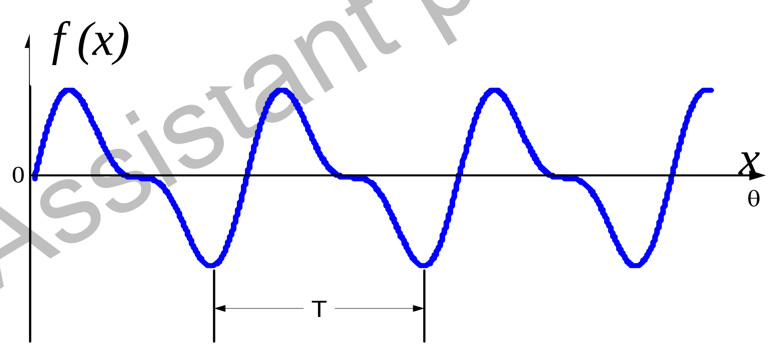
Fourier proposed in 1807 a periodic waveform *f(t)* could be broken down into an *infinite series of simple sinusoids* which, when added together, would construct the *exact form* of the original waveform.

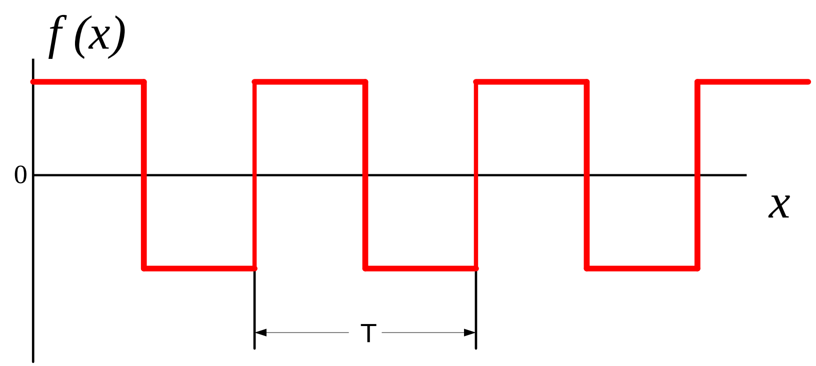
Fourier series are infinite series that represent periodic functions in terms of cosines and sines.

Periodic Functions

A **periodic function** is a **function** that repeats its values at regular intervals. A function $f(x)$ is said to be periodic function with period $T > 0$ if for all x , $f(x+T) = f(x)$, and T is the least of such values.

Ex: 1) $\sin x$, $\cos x$ are periodic functions with period 2π .





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Consider the periodic function $f(t) = f(t + nT) \quad ; \; n = \pm 1, \pm 2, \pm 3, \ldots$

$T =$ Period, the smallest value of T that satisfies the above Equation.

Then the *Fourier series* representation of *f* is a trigonometric series (sine and cosine terms)

$$f(t) = \frac{a_0}{2} + \underbrace{\sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T}}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}}_{\text{Odd Part}}$$

T is a period of all the above signals

Suppose *f* is a periodic function with a period $T = 2L$. Then the *Fourier series* representation of *f* is a trigonometric series (that is, it is an infinite series consists of sine and cosine terms) of the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

Where the coefficients are given by

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \qquad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

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Definition of Fourier series

1) Let *f(x)* be a function defined in **(0, 2π)** . Let *f(x+ 2π)= f(x)*, then the Fourier Series of *f(x) is given by*

f(x) is a periodic function;

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\ cosnx + b_n \sin nx\ )$$

Where a_0, a_n and b_n are constants.

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Where a_0, a_n and b_n are called as Fourier coefficient of *f(x)* in **(0, 2π).**

2) Let *f(x)* be a function defined in **(- π, π).** Let *f(x+ 2π)= f(x)*, then the Fourier Series of *f(x) is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\ cosnx + b_n \sin nx\ )$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Where a_0, a_n and b_n are as Fourier coefficient of *f(x)* in **(- π, π)**

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3) **Let *f(x)* be a function defined in (0, 2*l*). Let *f(x+ 2l)= f(x)*, then the Fourier Series of *f(x) is given by***

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx \qquad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx \qquad b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Where a_0, a_n and b_n are as Fourier coefficient of *f*(*x*) in **(0, 2*l*).**

Let *f(x)* be a function defined in **(- *l*, *l*).** Let *f(x+ 2l)= f(x)*, then the Fourier Series of *f(x) is given by*

where

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx \qquad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Where a_0, a_n and b_n are as Fourier coefficient of *f*(*x*) in **(- *l*, *l*).**

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}$$

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1) Expand the Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$

Solution : We recall the Fourier series of $f(x)$ that defined in the interval $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Here $f(x) = x^2$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{3\pi}[(2\pi)^3 - 0] = \frac{8}{3}\pi^2$$

$$a_0 = \frac{8}{3}\pi^2$$

$$\int x^6 \cdot e^x\,dx$$

D		I
x^6	+	e^x
$6x^5$	−	e^x
$30x^4$	+	e^x
$120x^3$	−	e^x
$360x^2$	+	e^x
$720x$	−	e^x
720	+	e^x
0		e^x

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\,\underbrace{\cos nx\,dx}_{dv}$$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{1}{\pi}\left[x^2\int \cos nx\,dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \cos nx\,dx\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \left\{\int 2x\left(\frac{\sin nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left\{\int \underbrace{x}_{u}\,\underbrace{\sin nx\,dx}_{dv}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \int 1.\frac{\cos nx}{n}dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\int \cos nx\,dx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \frac{2}{n}\left(-x\frac{\cos nx}{n} + \frac{1}{n}\frac{\sin nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + \frac{2}{n^2}x\cos nx - \frac{2}{n^3}\sin nx\right]_0^{2\pi}$$

$$= \frac{4}{n^2} \qquad \begin{matrix}\cos 2n\pi = 1 \\ \sin 2n\pi = 0\end{matrix}$$

$$a_n = \frac{4}{n^2}$$

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$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x^2}_{u}\ \underbrace{\sin nx}_{dv}\ dx$$

$$\left[\because \int uv\,dx = u\int v\,dx - \left\{\int \frac{du}{dx}.\left(\int v\,dx\right)dx\right\}\right]$$

$$\int u\,dv = uv - \int v\,du$$

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$$= \frac{1}{\pi}\left[x^2\int \sin nx\,dx - \left\{\int \frac{d}{dx}(x^2)\left(\int \sin nx\,dx\right)dx\right\}\right]$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - \left\{\int 2x\left(-\frac{\cos nx}{n}\right)dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left\{\int \underbrace{x}_{u}\ \underbrace{\cos nx}_{d\,v}\ dx\right\}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \int 1.\frac{\sin nx}{n}dx\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} - \frac{1}{n}\int \sin nx\,dx\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n}\left(x\frac{\sin nx}{n} + \frac{1}{n}\frac{\cos nx}{n}\right)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[-x^2\left(\frac{\cos nx}{n}\right) + \frac{2}{n^2}\,x\sin nx + \frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

$$b_n = -\frac{4\pi}{n}$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x^2)\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{4\pi^2}{n} + 0 + \frac{2}{n^3}\right\} - \left\{0 + 0 + \frac{2}{n^3}\right\}\right]$$

$$= -\frac{4\pi}{n}$$

D		I
x^6	+	e^x
$6x^5$	−	e^x
$30x^4$	+	e^x
$120x^3$	−	e^x
$360x^2$	+	e^x
$720x$	−	e^x
720	+	e^x
0		e^x

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$f(x) = \frac{\frac{8}{3}\pi^2}{2} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

$$f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

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Example 2 Expand in Fourier series of $f(x) = x \sin x$ for $0 < x < 2\pi$:

Sol. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_o = \frac{1}{\pi}\int_0^{2\pi} \underbrace{x}_{u}\ \underbrace{\sin x\,dx}_{dv}$$

$$\int u\,dv = uv - \int v\,du$$

$$= \frac{1}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[(-2\pi + 0) - (0 + 0)\right] \quad = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x(\cos nx\sin x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx\ , \quad n \neq 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin(n+1)x\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\sin(n-1)x\,dx$$

Product to Sum Formulas

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

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$$= \frac{1}{2\pi}\left[(x)\left(\frac{-\cos(n+1)x}{n+1}\right)-(1)\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\left[(x)\left(\frac{-\cos(n-1)x}{n-1}\right)-(1)\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[\left\{\frac{-2\pi(-1)^{2n+2}}{n+1}+0\right\}-\{0+0\}\right]-\frac{1}{2\pi}\left[\left\{\frac{-2\pi(-1)^{2n-2}}{n-1}+0\right\}-\{0+0\}\right]$$

$$= \frac{-1}{n+1}+\frac{1}{n-1}$$

$$a_n = \frac{-(n-1)+(n+1)}{(n+1)(n-1)}$$

$$a_n=\frac{2}{n^2-1} \quad n \neq 1$$

Note: $(-1)^{2n+2} = 1$

$(-1)^{2n-2} = 1$

When n = 1, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x\, dx$$

$$= \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-(1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi\left(\frac{-1}{2}\right)+0\right\}-(0+0)\right]$$

$$= -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int^{2\pi} x\sin x\sin nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin nx\sin x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x-\cos(n+1)x]\, dx, \quad n\neq 1$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\cos(n-1)x\, dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos(n+1)x\, dx$$

$$= \frac{1}{2\pi}\left[(x)\left(\frac{\sin(n-1)x}{n-1}\right)-(1)\left(\frac{-\cos(n-1)x}{(n-1)^2}\right)\right]_0^{2\pi}$$

$$-\frac{1}{2\pi}\left[(x)\left(\frac{\sin(n+1)x}{n+1}\right)-(1)\left(\frac{-\cos(n+1)x}{(n+1)^2}\right)\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[\left\{0+\frac{(-1)^{2n-2}}{(n-1)^2}\right\}-\left\{0+\frac{1}{(n-1)^2}\right\}\right]-\frac{1}{2\pi}\left[\left\{0+\frac{(-1)^{2n+2}}{(n+1)^2}\right\}-\left\{0+\frac{1}{(n+1)^2}\right\}\right]$$

$$=\frac{1}{2\pi}\left[\left\{0+\frac{1}{(n-1)^2}\right\}-\left\{0+\frac{1}{(n-1)^2}\right\}\right]-\frac{1}{2\pi}\left[\left\{0+\frac{1}{(n+1)^2}\right\}-\left\{0+\frac{1}{(n+1)^2}\right\}\right]$$

$$b_n = 0 \ , \quad n \neq 1$$

When n = 1, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin^2 x\,dx = \frac{1}{\pi}\int_0^{2\pi} x\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2\pi}\left[\frac{x^2}{2}-\left\{x\left(\frac{\sin 2x}{2}\right)-(1)\left(\frac{-\cos 2x}{4}\right)\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi^2-0-\left(\frac{1}{4}\right)\right\}-\left\{0-0-\frac{1}{4}\right]\right.$$

$$= \pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\, cosnx + b_n \sin nx\,)$$

$$= \frac{a_0}{2} + a_1\cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= \frac{-2}{2} - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2}{(n-1)(n+1)}\cos nx + \pi\sin x + 0$$

Example 3) Expand $f(x) = x - x^2$ as a Fourier series in $-l < x < l$.

Solution

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$\sin(-n\pi) = -\sin(n\pi) = 0$ for all integer n

$\cos(-n\pi) = \cos(n\pi) = (-1)^n$ for all integer n

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l}(x - x^2)\,dx$$

$$= \frac{1}{l}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{\frac{l^2}{2} - \frac{l^3}{3}\right\} - \left\{\frac{l^2}{2} + \frac{l^3}{3}\right\}\right] = \frac{1}{l}\left(\frac{-2l^3}{3}\right) = \frac{-2l^2}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l}(x - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[(x - x^2)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1 - 2x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{0 + (1-2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right) + 0\right\} - \left\{0 + (1+2l)\left(\frac{(-1)^n l^2}{n^2\pi^2}\right) + 0\right\}\right]$$

$$= \frac{(-1)^n l^2}{l\, n^2\pi^2}\left[1 - 2l - 1 - 2l\right] = \frac{(-1)^n l}{n^2\pi^2}\left[-4l\right]$$

$$a_n = \frac{4\,l^2(-1)^{n+1}}{n^2\,\pi^2}$$

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$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{1}{l}\int_{-l}^{l}(x - x^2)\sin\frac{n\pi x}{l}dx$$

$$= \frac{1}{l}\left[(x-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (1-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\left\{-(l-l^2)\left(\frac{(-1)^n l}{n\pi}\right) + 0 - \frac{2(-1)^n l^3}{n^3\pi^3}\right\} - \left\{-(-l-l^2)\left(\frac{(-1)^n l}{n\pi}\right) + 0 - \frac{2(-1)^n l^3}{n^3\pi^3}\right\}\right]$$

$$= \frac{-(-1)^n l}{l\,n\pi}\left[l - l^2 + l + l^2\right]$$

$$= \frac{(-1)^{n+1}}{n\pi}[2l]$$

$$\boxed{= \frac{2l\,(-1)^{n+1}}{n\pi}}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2}\left(\frac{-2l^2}{3}\right) + \sum_{n=1}^{\infty}\left(\frac{4l^2(-1)^{n+1}}{n^2\pi^2}\cos\frac{n\pi x}{l} + \frac{2l(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{l}\right)$$

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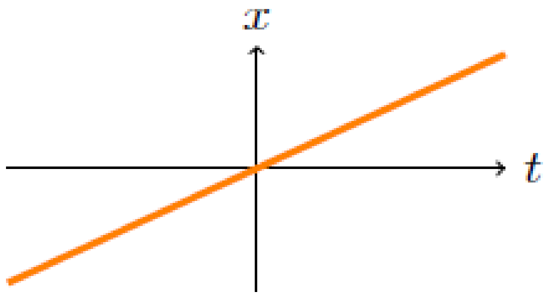
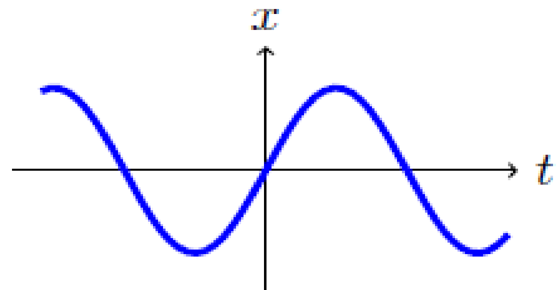
Odd Function: A function $F(x)$ is called an odd function if $F(-x) = -F(x)$

The graph of an **odd** function is always symmetrical about the **origin**



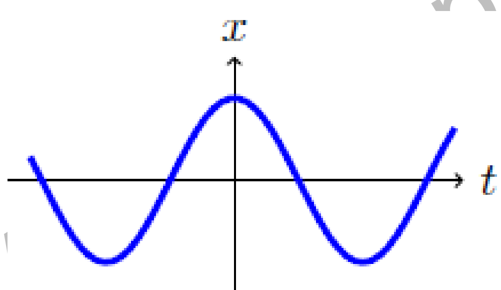


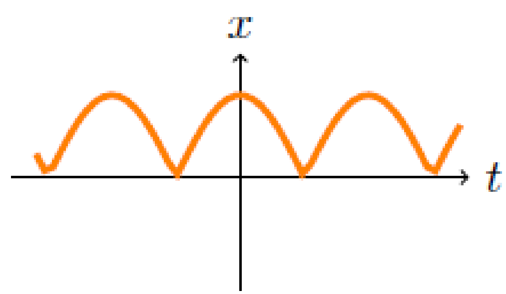
Example: x^3, $sinx$, $tanx$, $-x^5$ etc.

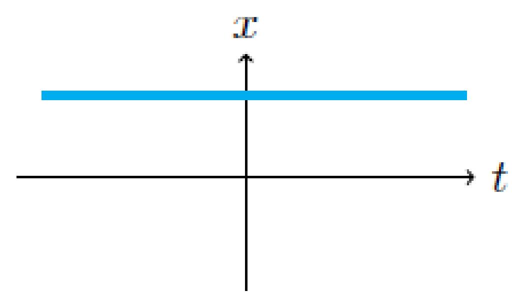
Products of functions
(even)×(even) = (even)
(even)×(odd) = (odd)
(odd)×(odd) = (even)

Even Function: A function $F(x)$ is called an even function if $F(-x) = F(x)$

The graph of an **even** function symmetric about the **y- axis**







Example: x^6, $\cos x$, $secx$,

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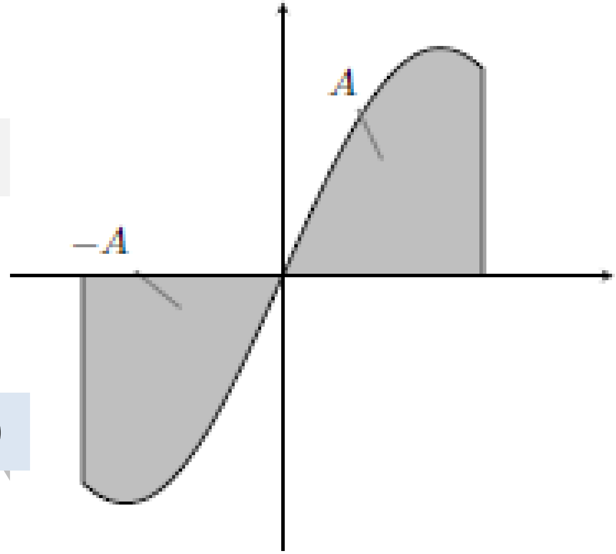
Even and Odd functions



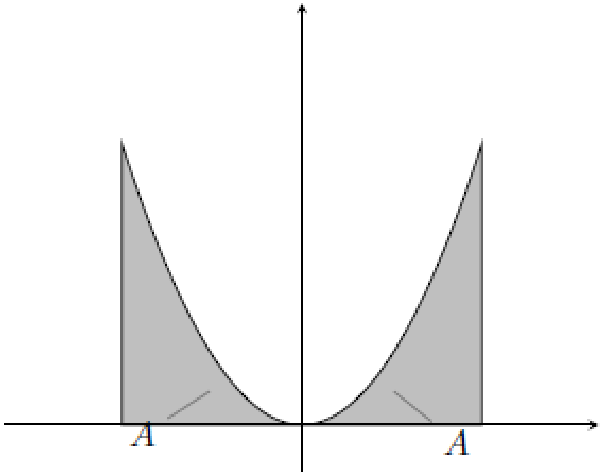
1) Let $p > 0$ be any fixed number. If $f(x)$ is an **odd function**, then

The integral of an odd function from $-p$ to p is zero

$$\begin{matrix} Area\ under\ f\ and\ to \\ the\ left\ of\ x = 0 \end{matrix} + \begin{matrix} Area\ under\ f\ and\ to\ the \\ right\ of\ x = 0 \end{matrix} = 0$$

$$\int_{-p}^{p} f(x)\, dx = 0.$$

if f is an odd function.



2) Let p > 0 be any fixed number. If $f(x)$ is an **even function**, then

The integral of an even function from $-p$ *to* p is twice the integral from 0 to p.

$$\begin{matrix} Area\ under\ f\ and\ to \\ the\ left\ of\ x = 0 \end{matrix} + \begin{matrix} area\ under\ f\ and\ to\ the \\ right\ of\ x = 0 \end{matrix}$$

= 2 [area under f and to the right of $x = 0$]

$$\int_{-p}^{p} f(x)\, dx = 2\int_{0}^{p} f(x)\, dx.$$

if f is an even function.

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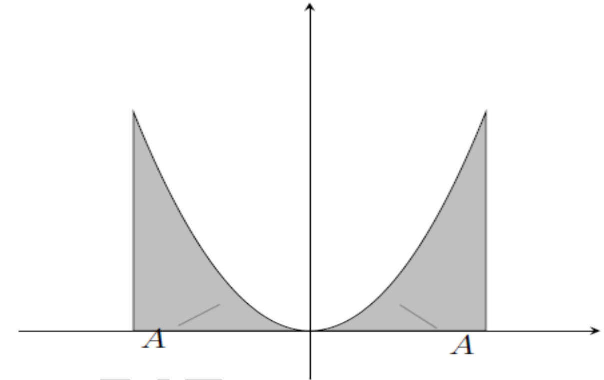
Fourier Series of Even Function



Let *f(x)* be a function defined in **(- *l*, *l*)**. Let *f(x+ 2l)= f(x)*, then the Fourier Series of *f(x) is given by*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx \qquad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx \qquad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx$$

Case 1) If $f(x)$ is an even function of period $2\,l$, then

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

Products of functions

(even)×(even) = (even)
(even)×(odd) = (odd)
(odd)×(odd) = (even)

Since $\cos nx$ is an even function, $f(x)$ is an even function

⟹ Product of two even functions is even

$$a_n = \frac{1}{l}\int_{-l}^{l} \underbrace{f(x)}_{even}\ \underbrace{\cos\frac{n\pi x}{l}}_{even} dx \quad = \quad even$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \overbrace{\underbrace{f(x)}_{\text{even}} \underbrace{\sin\left(\frac{n\pi x}{p}\right)}_{\text{odd}}}^{\text{odd}}\, dx = 0$$

(i) $a_0 = \frac{2}{l}\int_0^l F(x)dx$

(ii) $a_n = \frac{2}{l}\int_0^l F(x)\cos\frac{n\pi x}{l} dx,\ n = 1,2,3,\ldots$ (iii) $b_n = 0$

If a periodic function *f(x)* is an even function we have already used the fact that its Fourier series will involve only cosines

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Case (2): Fourier Series of Odd Function

(even)×(odd) = (odd)
(odd)×(odd) = (even)

If $F(x)$ is an odd function of period $2l$, then

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

If *f(x)* is odd, then we get

$$a_0 = \frac{1}{p}\int_{-p}^{p} \overbrace{f(x)}^{\text{odd}}\, dx = 0$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \overbrace{\underbrace{f(x)}_{\text{odd}} \underbrace{\cos(\frac{n\pi x}{p})}_{\text{even}}}^{\text{odd}}\, dx = 0$$

Where, (i) $a_0 = 0$ (ii) $a_n = 0$

(iii) $b_n = \frac{2}{l}\int_0^l \underbrace{F(x)}_{odd}\, \underbrace{\sin\frac{n\pi x}{l}}_{odd}\, dx, n = 1, 2, 3 \ldots$ $= even$

$\sin\frac{n\pi x}{l}$ is an odd function, *f(x)* is an odd function Product of two odd functions is even

Thus, if a function *f(x)* is Odd in *(-l, l)* , its Fourier series expansion contains only sine terms.

Half Range Series

The Fourier series which contains terms of sine or cosine only is known as half range Fourier sine series or half range Fourier cosine series.

Case-1 Half range Fourier cosine series:

For the half range Fourier cosine series of the function *f(x)* in the range *(0,l)*, we extend the function *f(x)* over the range *(-l, l)*. So that the function become even function.

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

(i) $a_0 = \frac{2}{l}\int_0^l F(x)dx$

(ii) $a_n = \frac{2}{l}\int_0^l F(x)\cos\frac{n\pi x}{l}\, dx, n = 1,2,3, \ldots$

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Half Range Fourier Cosine Series defined in $(0, \pi)$: The Fourier half range Cosine series in $(0, \pi)$ is given by

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\, nx$$

(i) $a_0 = \frac{2}{\pi}\int_0^{\pi} F(x)dx$ (ii) $a_n = \frac{2}{\pi}\int_0^{\pi} F(x)\cos\, nx\, dx\,, \quad n = 1,2,3, \ldots$

This is Similar to the Fourier series defined for even function in $(-\pi, \pi)$.

Case-2 Half range Fourier sine series:

For half range Fourier sine series of function *f(x)*,in the range*(0,l)*, we extend the function *f(x)* over the range *(-l, l);* so that the function becomes odd function.

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad b_n = \frac{2}{l}\int_0^{l} F(x)\sin\frac{n\pi x}{l}dx\,, n = 1,2,3 \ldots$$

Half Range Fourier Sine Series defined in $(0, \pi)$: The Fourier half range sine series in $(0, \pi)$ is given by

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad where \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx\,, \qquad n = 1,2,3 \ldots$$

This is Similar to the Fourier series defined for odd function in $(-\pi, \pi)$.

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Example Find the Fourier Series of the $F(x) = x, -\pi \le x \le \pi$

Solution: Given, $F(x) = x, Here, F(-x) = -x = -F(x)$

Therefore, $F(x)$ is an odd function. We have Fourier Series of Odd function

$$F(x) = \sum_{n=1}^{\infty} b_n sinnx$$

$$\text{Here, } b_n = \frac{2}{\pi}\int_0^{\pi} F(x)\ \sin\frac{n\pi x}{\pi}\,dx = \frac{2}{\pi}\int_0^{\pi} x\ \sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \int 1\left(-\frac{\cos nx}{n}\right)dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi} = \frac{2}{\pi}\left[-\frac{\pi\cos n\pi}{n}\right]$$

$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$[\because \sin n\pi = 0 \text{ and } \cos n\pi = (-1)^n]$$

Now from (i) we get,

$$F(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} sinnx$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

$$= 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$$

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Example : Find the Fourier Series of the $F(x) = x^2, -\pi \le x \le \pi$

Solution: Given, $F(x) = x^2, Here, F(-x) = (-x)^2 = F(x)$

Therefore, $F(x)$ is an even function. We have Fourier Series of Even function

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \dots\dots\dots (1)$$

Here $a_0 = \frac{2}{\pi}\int_0^{\pi} F(x)dx$ $= \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2\pi^3}{3} \quad \dots\dots\dots(ii)$

(ii) $a_n = \frac{2}{\pi}\int_0^{\pi} F(x) \cos\ nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - \int 2x\left(\frac{\sin nx}{n}\right)dx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi}{n^2}\cos n\pi + 0\right] = \frac{4}{n^2}(-1)^n \quad \dots\dots(iii)$$

Now from (i) we get,

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\ (-)^n \cos nx$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \cdots\right]$$

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Example: $F(x) = x, 0 \le x \le 2$ *in a half sine series*

Solution: We have, the half range Fourier sine series,

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_n = 0 \text{ and } b_n = \frac{2}{l}\int_0^l F(x)\ \sin\frac{n\pi x}{l}dx$$

where, $a_0 = 0$ $a_n = 0$ $b_n = \frac{2}{l}\int_0^l F(x)\sin\frac{n\pi x}{l}dx$

$$b_n = \frac{2}{2}\int_0^2 x\,\sin\frac{n\pi x}{2}dx$$

$$= \left[2x\left(-\frac{1}{n\pi}\cos\frac{n\pi x}{2}\right) - \int 2\left(-\frac{1}{n\pi}\cos\frac{n\pi x}{2}\right)dx\right]_0^2$$

$$= 2\left[-\frac{x}{n\pi}\cos\frac{n\pi x}{2} + \frac{2}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_0^2$$

$$= 2\left[-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right] = -\frac{4}{n\pi}\cos n\pi$$

$$[\because \sin n\pi = 0 \text{ and } \cos n\pi = (-1)^n]$$

$$= -\frac{4}{n\pi}(-1)^n$$

Now , we get,

$$F(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\ (-1)^n \sin nx$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\pi x + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots \ldots \ldots ..\right]$$

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Example **Find the half range sine series of $f(x) = x\cos x$ in $(0, \pi)$.**

Sol. Fourier sine series is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\,\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,(2\sin nx\cos x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]\,dx\ , \quad n \neq 1$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx\ , \quad n \neq 1$$

$$b_n = \frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[x\left(\frac{-\cos(n-1)x}{n-1}\right) - (1)\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\frac{-\pi(-1)^{n+1}}{n+1} + 0\right\} - \{0+0\}\right] + \frac{1}{\pi}\left[\left\{\frac{-\pi(-1)^{n-1}}{n-1} + 0\right\} - \{0+0\}\right]$$

$$= \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^n}{n-1} = (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right] = (-1)^n\left[\frac{2n}{(n+1)(n-1)}\right] \qquad (i.e.)\ b_n = \frac{2n(-1)^n}{n^2-1}\ , \quad n \neq 1$$

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When n = 1, we have

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin x\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left\{\pi\left(\frac{-1}{2}\right) + 0\right\} - \{0+0\}\right] = -\frac{1}{2}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -\frac{1}{2}\sin x + \sum_{n=2}^{\infty}\frac{2n(-1)^n}{n^2-1}\sin nx$$

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Example **Find the half range cosine series for the function $f(x) = x(\pi - x)$ in $0 < x < \pi$.**

Sol. Half range fourier cosine series is

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) - (0-0)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{6}\right] \qquad = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 + \frac{(-\pi)(-1)^n}{n^2} + 0\right\} - \left\{0 + \frac{(\pi)(1)}{n^2} + 0\right\}\right]$$

$$= \frac{2\pi}{\pi n^2}\left[-(-1)^n - 1\right]$$

$$= -\frac{2}{n^2}[(-1)^n + 1]$$

$$a_n = -\frac{4}{n^2}, \text{ when } n \text{ is even}$$

$$= 0 \quad , \text{ when } n \text{ is odd}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{1}{2}\left(\frac{\pi^2}{3}\right) + \sum_{n=even}^{\infty} -\frac{4}{n^2}\cos nx$$

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Complex Fourier Series

The Complex Fourier Series is the Fourier Series but written using e^{ix}

In order to derive the complex Fourier series, we first recall from last lecture the trigonometric Fourier series representation of a function defined on *[-l, l]* with period 2 π. The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where the Fourier coefficients were found as

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

In order to derive the exponential Fourier series, we replace the trigonometric functions with exponential functions and collect like exponential terms.

$$\cos\frac{n\pi}{l}x = \frac{e^{i\frac{n\pi x}{l}} + e^{-i\frac{n\pi x}{l}}}{2}$$

$$\sin\frac{n\pi}{l}x = \frac{e^{i\frac{n\pi x}{l}} - e^{-i\frac{n\pi x}{l}}}{2i}$$

This gives

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\left(\frac{e^{i\frac{n\pi x}{l}} + e^{-i\frac{n\pi x}{l}}}{2}\right) + b_n\left(\frac{e^{i\frac{n\pi x}{l}} - e^{-i\frac{n\pi x}{l}}}{2i}\right)\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - ib_n}{2}\right)e^{i\frac{n\pi x}{l}}\right] + \sum_{n=1}^{\infty}\left[\left(\frac{a_n + i\,b_n}{2}\right)e^{-i\frac{n\pi x}{l}}\right]$$

The coefficients of the complex exponentials can be rewritten by defining

$$c_n = \frac{a_n - i\,b_n}{2} \qquad \bar{c}_n = \frac{a_n + i\,b_n}{2}$$

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So far, the representation is rewritten as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(c_n \quad e^{i\frac{n\pi x}{l}}\right) + \sum_{n=1}^{\infty}\left(\bar{c}_n \quad e^{-i\frac{n\pi x}{l}}\right)$$

Re-indexing the first sum, by introducing k = - n, we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(c_n \quad e^{i\frac{n\pi x}{l}}\right) + \sum_{k=-1}^{-\infty}\left(\bar{c}_{-k} \quad e^{i\frac{k\pi x}{l}}\right)$$

Since k is a dummy index, we replace it with a new n as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(c_n \quad e^{i\frac{n\pi x}{l}}\right) + \sum_{n=-1}^{-\infty}\left(\bar{c}_{-n} \quad e^{i\frac{n\pi x}{l}}\right)$$

$$\bar{c}_n = \frac{a_n + i\, b_n}{2}$$

$$a_{-n} = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{-n\pi x}{l}\,dx = a_n$$

$$b_{-n} = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{-n\pi x}{l}\,dx = -b_n$$

$$c_n = \bar{c}_{-n} = \frac{a_n - i\, b_n}{2}$$

Letting $c_0 = \frac{a_0}{2}$ we can write the complex exponential Fourier series representation as

Complex Fourier Series for a function of period **2*l***

$$f(x) = \sum_{n=-\infty}^{\infty}\left(c_n \quad e^{i\frac{n\pi x}{l}}\right)$$

Complex Fourier Series for a function of period **2π**: $f(x) = \sum_{n=-\infty}^{\infty}\left(c_n \quad e^{inx}\right)$

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Given such a representation, we would like to write out the integral forms of the coefficients, c_n. So, we replace the a_n's and b_n's with their integral representations and replace the trigonometric functions with complex exponential functions. Doing this, we have for n = 1, 2, . . . ,

$$c_n = \frac{a_n - i\,b_n}{2}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\,(\cos\frac{n\pi x}{l}\,dx - i\,\frac{1}{2l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\,(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\,)\,dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\,e^{-i\frac{n\pi x}{l}}\,dx$$

It is a simple matter to determine the c_n's for other values of n. For n = 0, we have that

$$c_0 = \frac{a_0}{2} = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

Complex Fourier Series for a function of period $2l$

$$f(x) = \sum_{n=-\infty}^{\infty}\left(c_n\ e^{i\frac{n\pi x}{l}}\right)$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\,e^{-i\frac{n\pi x}{l}}\,dx$$

Complex Fourier Series for a function of period 2π:

$$f(x) = \sum_{n=-\infty}^{\infty}\left(c_n\ e^{inx}\right)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx$$

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Example Find the complex Fourier series for the function $f(x) = e^x$ on the interval [-1, 1]

solution

$$f(x) = \sum_{n=-\infty}^{\infty} \left(c_n \; e^{i\frac{n\pi x}{l}}\right)$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-i\frac{n\pi x}{l}}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^x e^{-in\pi x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{(1-in\pi)x}\, dx$$

$$= \frac{1}{2(1-in\pi)}\left[e^{1-in\pi} - e^{-1+in\pi}\right]$$

$$= \frac{(-1)^n}{2(1-in\pi)}(e - 1/e).$$

The last identity follows since $e^{in\pi} = e^{-in\pi} = (-1)^n$.

The complex Fourier series is

$$e^x \sim \frac{e - 1/e}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1 - in\pi} e^{in\pi x} \quad \text{for } -1 \le x \le 1.$$

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Example

Using complex form, find the Fourier series of the function

$$f(x) = \operatorname{sign} x = \begin{cases} -1, & -\pi \le x \le 0 \\ 1, & 0 < x \le \pi \end{cases}$$

Solution

$$f(x) = \sum_{n=-\infty}^{\infty} \left(c_n \; e^{inx}\right)$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

We calculate the coefficients c_0 and c_n for

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi} dx\right] = \frac{1}{2\pi}\left[(-x)\Big|_{-\pi}^{0} + x\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}(-\pi + \pi) = 0,$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,e^{-inx}dx + \int_{0}^{\pi} e^{-inx}dx\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\left(e^{-inx}\right)\Big|_{-\pi}^{0}}{-in} + \frac{\left(e^{-inx}\right)\Big|_{0}^{\pi}}{-in}\right] = \frac{i}{2\pi n}\left[-\left(1 - e^{in\pi}\right) + e^{-in\pi} - 1\right]$$

$$= \frac{i}{2\pi n}\left[e^{in\pi} + e^{-in\pi} - 2\right] = \frac{i}{\pi n}\left[\frac{e^{in\pi} + e^{-in\pi}}{2} - 1\right] = \frac{i}{\pi n}\left[\cos n\pi - 1\right]$$

$$c_n = \frac{i}{\pi n}\left[(-1)^n - 1\right]$$

$$f(x) = \sum_{n=-\infty}^{\infty} \left(c_n \; e^{inx}\right)$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{i}{\pi n}\left[(-1)^n - 1\right] e^{inx}$$

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Example Find the complex Fourier series for $f(x) = x$ in (-2 , 2).

solution $f(x) = \sum_{n=-\infty}^{\infty} \left(c_n \quad e^{i\frac{n\pi x}{l}}\right)$ $c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-i\frac{n\pi x}{l}}\, dx$

We can write $c_n = \frac{1}{4}\int_{-2}^{2} x e^{-\pi i n x/2} dx$

Integration by parts $\int u\,dv = uv - \int v\,du$ with $u = x$ and $dv = e^{-\pi i n x/2}dx$ so $du = dx$ and $v = \frac{-2}{\pi i n} e^{-\pi i n x/2}$

$$c_n = \frac{1}{4}\left[\frac{-2x}{\pi i n} e^{-\pi i n x/2} + \int \frac{2}{\pi i n} e^{-\pi i n x/2} dx\right]_{-2}^{2} = \frac{1}{4}\left[\frac{-2x}{\pi i n} e^{-\pi i n x/2} - \frac{4}{\pi^2 i^2 n^2} e^{-\pi i n x/2}\right]_{-2}^{2} = \left[\frac{-x}{2\pi i n} e^{-\pi i n x/2} + \frac{1}{\pi^2 n^2} e^{-\pi i n x/2}\right]_{-2}^{2}$$

$$C_n = \left[\frac{-1}{\pi i n} e^{-\pi i n} + \frac{1}{\pi^2 n^2} e^{-\pi i n}\right] - \left[\frac{1}{\pi i n} e^{\pi i n} + \frac{1}{\pi^2 n^2} e^{\pi i n}\right] = \frac{-1}{\pi i n}\left(e^{-\pi i n} + e^{\pi i n}\right) + \frac{1}{\pi^2 n^2}\left(e^{-\pi i n} - e^{\pi i n}\right)$$

Since $\frac{-1}{i} \times \frac{i}{i} = i$ then $C_n = \frac{i}{\pi n}\left(e^{-\pi i n} + e^{\pi i n}\right) + \frac{1}{\pi^2 n^2}\left(e^{-\pi i n} - e^{\pi i n}\right)$

It is known that since $e^{\pi i n} = \cos n\pi + i \sin n\pi$ and $e^{-\pi i n} = \cos n\pi - i \sin n\pi$ then ……

$\cos n\pi = \frac{1}{2}\left(e^{-\pi i n} + e^{\pi i n}\right)$ and $\sin n\pi = \frac{-1}{2i}\left(e^{-\pi i n} - e^{\pi i n}\right)$ so we say $C_n = \frac{2i}{\pi n}\cos n\pi - \frac{2i}{\pi^2 n^2}\sin n\pi = \frac{2i}{\pi n}\cos n\pi$

$C_n = \frac{2i}{\pi n}\cos n\pi = \frac{2i}{\pi n}(-1)^n$ then $f(x) = \sum_{n=-\infty}^{\infty} \frac{2i}{\pi n}(-1)^n e^{\pi i n x/2}$

Assistant Prof. Dr. Eng. Ibrahim Thamer Nazzal

Example

Find the complex form of the Fourier series of $f(x)$ = sin x in (0 , π).

Solution:

Here $2l = \pi \quad or \quad l = \pi / 2$.

The complex form of Fourier series is $f(x) = \sum_{n=-\infty}^{\infty} c_n \; e^{i2\,nx}$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x)\, e^{-i\frac{n\pi x}{l}}\, dx$$

$$c_n = \frac{1}{\pi}\int_0^{\pi} \sin x e^{-i2nx}\, dx$$

$$= \frac{1}{\pi}\left[\frac{e^{-i2nx}}{1-4n^2}\{-i2n\sin x - \cos x\}\right]_0^{\pi}$$

$$= \frac{1}{\pi(4n^2-1)}\left[-e^{i2nx} - 1\right] = -\frac{2}{\pi(4n^2-1)}$$

Using this value, we get

$$\sin x = -\frac{2}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{4n^2-1}.e^{i2nx} \qquad \text{in } (0\ ,\ \pi)$$

Assistant Prof. Dr. Eng. Ibrahim Thamer Nazzal